PHYS 705: Classical Mechanics

Notes:

- September 7 next Monday (Labor Day)
- Not all problems will be corrected. Check online solution!

LECTURE REVIEW

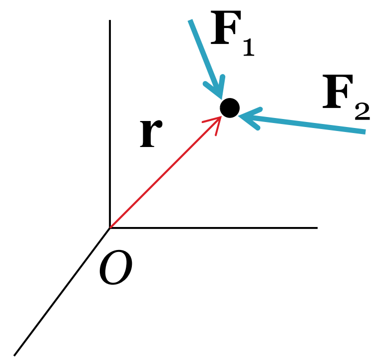
Newtonian Mechanics: Basic Description

Newton's second law of motion:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \dot{\mathbf{p}}$$

where $\mathbf{F} = \sum_i \mathbf{F}_i$ is the net sum (vector sum) of all forces acting on the particle

The influence of the external world is encoded as forces (vectors) $\mathbf{F}$ acting on the particle.



What we get:

Trajectory in configuration space given by the Newton's 2nd law !

Mechanics of a System of Particles

- For a system of particles, one needs to distinguish between :

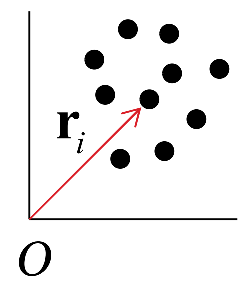
"external forces" acting *on* the entire system and
"internal forces" acting *within* the system

- 2nd law for the *i*th particle is:

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = \dot{\mathbf{p}}_i$$

internal force on i from j

net external force on i



2nd Law & Conservation Theorems for a System of Particles

1. If the weak form of Newton's 3rd law applies ..., i.e.,

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}$$

$$\boxed{\mathbf{F}_{tot}^{ext} = \dot{\mathbf{p}}_{tot}}$$

2nd Law for a System of Particles

If $\mathbf{F}_{tot}^{ext} = 0$, then $\dfrac{d\mathbf{p}_{tot}}{dt} = 0 \quad \Rightarrow \quad \mathbf{p}_{tot} = \text{constant}$

Conservation of Linear Momentum

2nd Law & Conservation Theorems for a System of Particles

2. If both the weak & strong forms of Newton's 3rd law apply …, i.e.,

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij} \qquad\qquad \mathbf{r}_{ij} \times \mathbf{F}_{ji} = 0$$

$$\Rightarrow \quad \frac{d\mathbf{L}_{tot}}{dt} = \mathbf{N}_{tot}^{ext}$$

2nd Law for angular variables (System of Particles)

If $\mathbf{N}_{tot}^{ext} = 0$, then $\dfrac{d\mathbf{L}_{tot}}{dt} = 0 \quad \Rightarrow \quad \mathbf{L}_{tot} = \text{constant}$

Conservation of Angular Momentum

2nd Law & Conservation Theorems for a System of Particles

3. If $\mathbf{F}$ is conservative, i.e., $\mathbf{F} = -\nabla U$

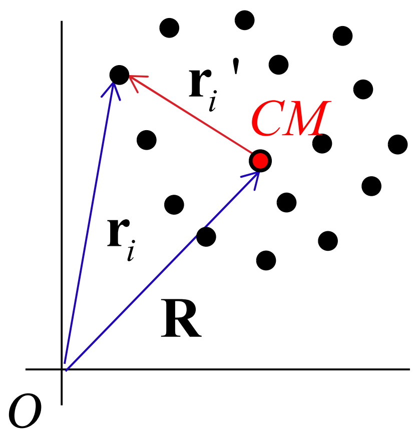
$$\Rightarrow \quad T_1 + U_1 = T_2 + U_2$$

Conservation of Mechanical Energy

where $T = \dfrac{1}{2} m v^2$ is the kinetic energy

General Motion of a System of Particles

Can be separated into: $\begin{pmatrix} \text{motion } \textit{of} \\ \text{the CM} \end{pmatrix} \oplus \begin{pmatrix} \text{motion } \textit{about} \\ \text{the CM} \end{pmatrix}$



$$\mathbf{R} + \mathbf{r}_i{}' = \mathbf{r}_i$$

$$\mathbf{V} + \mathbf{v}_i{}' = \mathbf{v}_i$$

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}$$

$$\mathbf{p}_{tot} = M\frac{d\mathbf{R}}{dt}$$

$$\mathbf{L}_{tot} = \mathbf{R} \times M\mathbf{V} + \sum_i \left(\mathbf{r}_i{}' \times m_i \mathbf{v}_i{}'\right)$$

$$T = \frac{1}{2}\sum_i m_i V^2 + \frac{1}{2}\sum_i m_i \left(v_i{}'\right)^2$$

$$U = U_{ext} + U_{\text{int}}$$

Holonomic Constraints

Holonomic constraints can be expressed as a function in terms of the coordinates and time,

$$f\left(\mathbf{r}_1, \mathbf{r}_2, \cdots; t\right) = 0$$

e.g. (a rigid body) → $\left(\mathbf{r}_i - \mathbf{r}_j\right)^2 - c_{ij}^2 = 0$

non-holonomic examples:
-Gas in a container
- Particle that slides then falls off a sphere: the constraint equation must be an inequality
- Object rolling on a rough surface without slipping... more later

More quantifiers:
- Rheonomous: depend on time explicitly
- Scleronomous: not explicitly depend on time

e.g. a bead constraints to move on a fixed vs. a moving wire

Generalized Coordinates

- Without constraints, a system of N particles has $3N$ dof
- With K constraint equations, the # dof reduces to $3N$-K
- With holonomic constraints, one can introduce ($3N$-K) **independent** (proper) generalized coordinates $(q_1, q_2, \cdots, q_{3N-K})$ such that:

$$\mathbf{r}_1 = \mathbf{r}_1(q_1, q_2, \cdots, q_{3N-K}, t)$$
$$\vdots$$
$$\mathbf{r}_N = \mathbf{r}_N(q_1, q_2, \cdots, q_{3N-K}, t)$$

➢ Generalized coordinates can be anything: angles, energy units, momentum units, or even amplitudes in the Fourier expansion of $\mathbf{r}_i$

➢ But, they must completely specify the state of a given system

➢The choice of a particular set of generalized coordinates is not unique.

➢No specific rule in finding the most "suitable" (resulting in simplest EOM)

D'Alembert's Principle

To formulate the mechanical problem with constraint forces so that they "disappear" → you solve the "new" problem using only the (given) applied forces.

Begin with the 2nd law, $\mathbf{F}_i = \dot{\mathbf{p}}_i$ or $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$

Consider a virtual infinitesimal displacement $\delta \mathbf{r}_i$ consistent with the given constraint,

$$\sum_i \left(\mathbf{F}_i - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0$$

Separating out the applied and constraint forces, $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$

This gives, $$\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

D'Alembert's Principle

$\begin{pmatrix} \text{Virtual displacement} \\ \text{to be consistent} \\ \text{w/ constraints} \end{pmatrix}$ means $\underbrace{\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0}_{\text{virtual work} = 0}$ $\Rightarrow$ $\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0$

This is the D'Alembert's Principle. Then, to solve for the EOM…

We need to look into changing the variables into a set of *independent* **generalized coordinates** so that we have

$$\sum_j (?)_j \cdot \delta q_j = 0$$

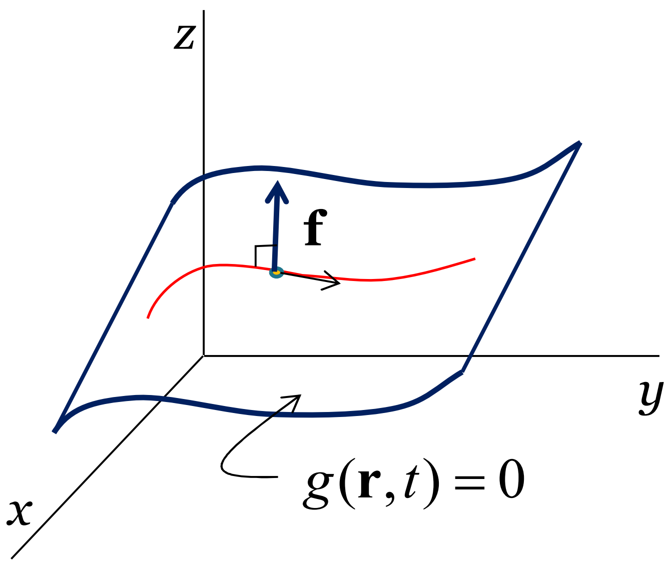
Then, we can claim the coefficients $(?)_j$ in the sum to be independently equal to zero and the **Euler-Lagrange equation** will give an explicit expression for the EOM as:

$$(?)_j = 0$$

Geometric View of the D'Alembert's Principle

$\begin{pmatrix} \text{Virtual displacement} \\ \text{to be consistent} \\ \text{w/ constraints} \end{pmatrix}$ or virtual work =0 or $\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$

➡ Constraint Force **f** needs to lay $\perp$ to the constraint surface



With $g(\mathbf{r},t)=0$ being the equation for the constraint surface and

$$\to \nabla g(\mathbf{r},t) \quad \perp \quad \text{surface}$$

We can "parametrized" **f** in term of $g(\mathbf{r},t)$,

$$\mathbf{f} = \lambda \nabla g(\mathbf{r},t) \qquad \text{where } \lambda \text{ is a parameter}$$

This gives,

$$\left.\begin{aligned} m\ddot{\mathbf{x}} &= \mathbf{F}^{(a)} + \lambda \nabla g(\mathbf{r},t) \\ g(\mathbf{r},t) &= 0 \end{aligned}\right\} \begin{array}{l} \text{4 unknowns } \mathbf{r} \text{ and } \lambda \\ \text{4 equations} \end{array}$$

Constraint and Work

Consider the EOM in this form: $m\ddot{\mathbf{r}} = \mathbf{F}^{(a)} + \lambda \nabla g(\mathbf{r}, t)$

Let $\mathbf{F}^{(a)}$ be a conservative force , i.e., $\mathbf{F}^{(a)} = -\nabla U(\mathbf{r}, t)$ so that

$$m\ddot{\mathbf{r}} = -\nabla U + \lambda \nabla g$$

Dotting $\dot{\mathbf{r}}$ into both sides,

$$m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{d}{dt}\left(\frac{1}{2} m\dot{\mathbf{r}}^2\right) = \frac{dT}{dt} \qquad\qquad -\nabla U \cdot \dot{\mathbf{r}} + \lambda \nabla g \cdot \dot{\mathbf{r}}$$

Consider the last term, from chain rule, we have,

$$\frac{dg}{dt} = \left(\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt}\right) + \frac{\partial g}{\partial t} = \left(\nabla g \cdot \dot{\mathbf{r}}\right) + \frac{\partial g}{\partial t}$$

Constraint and Work

As the particle moves, it is constrained to stay on the g=0 surface,

So, $\frac{dg}{dt} = 0$ and, $\left(\nabla g \cdot \dot{\mathbf{r}}\right) = -\frac{\partial g}{\partial t}$

Similarly, from chain rule, we can write, $\nabla U \cdot \dot{\mathbf{r}} = \frac{dU}{dt} - \frac{\partial U}{\partial t}$

Putting everything together,

$$m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -\nabla U \cdot \dot{\mathbf{r}} + \lambda \nabla g \cdot \dot{\mathbf{r}}$$

$$\frac{dT}{dt} = -\frac{dU}{dt} + \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}$$

With $E=T+U$,

$$\Rightarrow \quad \frac{dE}{dt} = \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}$$

Constraint and Work

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}$$

So, either U or g *explicitly* depends on time, the total energy will not be a constant in time.

Since we typically do not consider time-dependent U potential functions, So, we can make the following assertions:

Scleronomous (g not explicitly depends on t) Holonomic Constraints:

$$\left(\nabla g \cdot \dot{\mathbf{r}}\right) = -\frac{\partial g}{\partial t} = 0$$ and constraint force won't do work!

Rheonomous (g explicitly depends on t) Holonomic Constraints:

$$\left(\nabla g \cdot \dot{\mathbf{r}}\right) = -\frac{\partial g}{\partial t} \neq 0$$ and constraint force can do work!